Definable sets in pseudo-finite fields

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Model theory and Applications

Goal: general

- Let T be a theory.
  Natural question: classify 0-definable sets up to 0-definable bijection.
  Write Def(T) for 0-definable sets of T.
- Suppose ϕ_1, ϕ_2 are given.
  To prove $\phi_1 \overset{1:1}{\to} \phi_2$: write down the bijection.
  To prove $\phi_1 \overset{1:1}{\not\to} \phi_2$: use *invariants*:
  map $f\colon \mathrm{Def}(T) \longrightarrow \{\text{any set}\}$ s.t. $\phi \overset{1:1}{\to} \psi \Rightarrow f(\phi) = f(\psi)$

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This talk:

- $\mathrm{Th}(\mathrm{PSF}_0) :=$ theory of pseudo-finite fields of characteristic 0
- Main goal: Find good invariants for $\mathrm{Th}(\mathrm{PSF}_0)$
- To start: work in fixed pseudo-finite field K
  Method: counting over finite fields
  - $\rightsquigarrow$ dimension, measure
  - $\rightsquigarrow$ even more
- Transfer results from $\mathrm{Th}(K)$ to $\mathrm{Th}(\mathrm{PSF}_0)$
- New method:
  $\rightsquigarrow$ results for $\mathrm{Th}(\mathrm{PSF}_0)$ which counting does not yield
  $\rightsquigarrow$ transfer counting results to theories where counting does not make sense

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ACF-case

Start with easier example: $K \models \mathrm{Th}(\mathrm{ACF}_0) :=$ alg. closed fields of characteristic 0.

- Quantifier elimination ⇒ any ϕ defines a "constructible" set.
- Well-defined invariants:
    - $\dim_{\mathrm{alg}} \overline{\phi(K)}$
    - #irreducible components of max. dim. of $\overline{\phi(K)}$

Can we get similar invariants if K is pseudo-finite (for arbitrary ϕ)?
(Is it true, for pseudo-finite K, that $K \overset{1:1}{\not\twoheadrightarrow} K^2$?)

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Idea: counting

Th(PSF$_0$) is not complete. Fix K pseudo-finite with char $K = 0$ and work in Th(K).

- ϕ ring formula. $\rightsquigarrow$ can count $\#\phi(\mathbb{F}_q)$.
- This is not an invariant, but "for almost all q in ultra-filter sense":
  - $K \equiv \prod_{\mathcal{U}} \mathbb{F}_q$ for $\mathcal{U}$ ultrafilter on prime powers.
  - $K \models \chi \iff \{q \mid \mathbb{F}_q \models \chi\} \in \mathcal{U}$
  - So: $K \models \psi\colon \phi_1 \overset{1:1}{\to} \phi_2 \Rightarrow \{q \mid \#\phi_1(\mathbb{F}_q) = \#\phi_2(\mathbb{F}_q)\} \in \mathcal{U}$
- Formally: we have counting-invariant

$$\begin{aligned} cnt_K\colon \ \mathrm{Def}(\mathrm{Th}(K)) &\longrightarrow \mathbb{N}^{\mathcal{U}} \\ \phi &\longmapsto (\#\phi(\mathbb{F}_q))_q \end{aligned}$$

- Example: $\mathbb{A}^1 \overset{1:1}{\not\to} \mathbb{A}^2$ (always-true-formulas in 1 resp. 2 variables)
- Example: $\mathbb{A}^1 \overset{1:1}{\not\to} \mathbb{A}^1 \setminus \{0\}$

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What should counting yield?

Does cnt_K see the dimension and number of components of ϕ if ϕ is algebraic?

Theorem (Lang, Weil)

Let V be absolutely irreducible. Then $\#V(\mathbb{F}_q) \approx q^{\dim V}$ for $q \gg 0$.
("$\approx$" is almost independent of V!)

- Yes, if ϕ defines a union of absolutely irreducible varieties: $\#\phi(\mathbb{F}_q) \approx \#\text{comp-of-max-dim}(\phi) \cdot q^{\dim_{\text{alg}} \phi}$.
- Hope: for ϕ arbitrary, there exist d, μ such that $\#\phi(\mathbb{F}_q) \approx \mu \cdot q^d$
- Call d *dimension* of ϕ and μ *measure* of ϕ.

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- Yes, if ϕ defines a union of absolutely irreducible varieties: $\#\phi(\mathbb{F}_q) \approx \#\text{comp-of-max-dim}(\phi) \cdot q^{\dim_{\text{alg}} \phi}$.
- Hope: for ϕ arbitrary, there exist d, μ such that $\#\phi(\mathbb{F}_q) \approx \mu \cdot q^d$
- Call d *dimension* of ϕ and μ *measure* of ϕ.

Dimension and measure: more precisely

More precisely:

- Consider $\mu_d := \lim_{q\to\infty} \frac{\#\phi(\mathbb{F}_q)}{q^d}$.
  If $0 < \mu_d < \infty$, then define $\dim_K \phi := d, \mu_K(\phi) := \mu_d$.
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- Problem: Limit does not need to exist.
  Example: $\phi = \{\pm\sqrt{10}\}$

$$\#\phi(\mathbb{F}_q) = \begin{cases} 0 & \text{for infinitely many } q \\ 2 & \text{for infinitely many } q \end{cases}$$

- So recall $K \equiv \prod_{\mathcal{U}} \mathbb{F}_q$ and use ultra-limit:

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In particular, this is an invariant for K.

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Measure: existence theorem

Problem if e.g. $\#\phi(\mathbb{F}_q) \approx \log q$.

For $\dim_K \phi, \mu_K(\phi)$ to be well defined, we need:

Theorem (Chatzidakis, van den Dries, Macintyre)

For any $\phi \neq \emptyset$, there exists $d \in \mathbb{N}$ such that $0 < \mu_d < \infty$, where

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Proof for quantifier-free formulas: use the theorem of Lang-Weil

- We already saw: ϕ union of absolutely irreducible varieties $\Rightarrow$ $\dim_K \phi =$ algebraic dimension of ϕ, $\mu_K(\phi) = \#$components of max. dim.
- If $\phi = \{(x, y) \mid y^2 = 2x^2\} = \{(x, \pm\sqrt{2}x)\}$:
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Measure: idea of proof 2

Generalize to arbitrary formulas: use:

Theorem (Almost quantifier-elimination)

For any ϕ, there exists ψ quantifier free such that for all $\overline{a} \in K$:
$K \models \phi(\overline{a}) \leftrightarrow \exists \overline{y} \psi(\overline{a}, \overline{y})$
And: There is an $N \in \mathbb{N}$ such that for all $\overline{a} \in \phi(K)$:
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Properties of dimension and measure

A lot of properties follow easily from the definition:

Lemma

$$\dim_K(\phi \mathbin{\dot\cup} \psi) = \max\{\dim_K \phi, \dim_K \psi\}$$

$$\mu_K(\phi \mathbin{\dot\cup} \psi) = \begin{cases} \mu_K(\phi) + \mu_K(\psi) & \textit{if } \dim_K \phi = \dim_K \psi \\ \mu_K(\phi) & \textit{if } \dim_K \phi > \dim_K \psi \end{cases}$$

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If $f\colon \phi \twoheadrightarrow \psi$ *is* $n : 1$, *then*

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Example: $f\colon K \to K, x \mapsto x^2$ is $2 : 1$ onto the set of squares, so $\mu_K(\{\text{squares}\}) = \frac{1}{2}$.

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Summary of non-bijections

Using cnt_K, dimension, and measure, we can prove many non-bijections:

- $\phi = \{(x, y) \mid x^2 = y^3\}$, $\psi = \{\text{squares}\}$
  $\mu_K(\phi) = 1$, $\mu_K(\psi) = \frac{1}{2} \Rightarrow \phi \overset{1:1}{\not\to} \psi$
- $\phi' = \mathbb{A}^2 \setminus \phi$,
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- Up to now: $T = \mathrm{Th}(K)$ where K is pseudo-finite, char $K = 0$.
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Different notions of dimension

These $\dim_K$, μ_K are indeed different:

- Recall example: $\phi = \{(x,y) \mid y^2 = 2x^2\}$:
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K pseudo-finite yields $\mathrm{Def}(\mathrm{Th}(\mathrm{PSF}_0)) \longrightarrow \mathrm{Def}(\mathrm{Th}(K)) \xrightarrow{cnt_K} \mathbb{N}^{\mathcal{U}}$.

But we may also count directly for $\mathrm{Th}(\mathrm{PSF}_0)$:

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- So define

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- We get $$cnt\colon \mathrm{Def}(\mathrm{Th}(\mathrm{PSF}_0)) \longrightarrow S \\ \phi \longmapsto (\#\phi(\mathbb{F}_q))_q$$
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K pseudo-finite yields $\mathrm{Def}(\mathrm{Th}(\mathrm{PSF}_0)) \longrightarrow \mathrm{Def}(\mathrm{Th}(K)) \xrightarrow{cnt_K} \mathbb{N}^{\mathcal{U}}$.
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- $\phi = \{\text{squares}\} \setminus \{0\}$, $\psi = \{\text{non-squares}\}$.
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Other theories

We construct θ in a more general setting.

- Recall:

$$K \models \mathrm{Th}(\mathrm{PSF}_0) \iff K \text{ perfect and PAC, } \mathrm{Gal}(\tilde{K}/K) \cong \hat{\mathbb{Z}}, \mathrm{char}\, K = 0$$

- Replace $\hat{\mathbb{Z}}$ by other group G.
- Is "$\mathrm{Gal}(\tilde{K}/K) \cong G$" first order? Yes if G is *bounded* $:\iff$ finite number of quotients of each fixed cardinality.
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There are maps between $\mathrm{Def}(T_G)$ for different G:

Theorem (H.)

$G_2 \subset G_1$ pro-finite, bounded, G_2 characteristic subgroup.

Then there exists $\theta\colon \mathrm{Def}(T_{G_2}) \to \mathrm{Def}(T_{G_1})$

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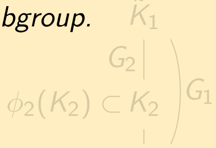

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Then there exists $\theta\colon \mathrm{Def}(T_{G_2}) \to \mathrm{Def}(T_{G_1})$

given by: $\theta(\phi_2) = \phi_1$ *where:*

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- Main statement: $\phi_2(K_2) \cap K_1$ is definable (uniformly in K_1).
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There are maps between $\mathrm{Def}(T_G)$ for different G:

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Squares $\overset{1:1}{\not\to}$ non-squares

- Let $T := \mathrm{Th}(\mathrm{PSF}_0)$.
- Recall example $\phi = \{\text{squares}\} \smallsetminus \{0\}$, $\psi = \{\text{non-squares}\}$.
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Invariants for T_G, $G \neq \hat{\mathbb{Z}}$

We know invariants for T_G if $G = \hat{\mathbb{Z}}$ or $G = \{1\}$. What about other G? If $G \subsetneq \hat{\mathbb{Z}}$:

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Idea of proof

For those who are interested: idea of proof of the theorem
(The others may sleep.)

- Recall: $G_2 \subset G_1$
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- Compatibility with definable bijections:
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Definability of $\phi_2(K_2) \cap K_1$

Definablility: shown on the example $G_1 = \hat{\mathbb{Z}}$ and $\phi_2 = \{\text{squares}\}$.

- $\phi_2(K_2)$ is image of K_2 under $f : x \mapsto x^2$.
  (In general: image of $V(K_2)$ under finite-to-1 map f.)
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