Approximate subgroups II

Freiman's inverse problem consists in characterizing those finite subsets X of a group G such that $X \cdot X$ is not much bigger than X. More precisely, assuming that $|X \cdot X| \leq K \cdot |X|$ for some fixed $K \in \mathbb{N}$, one would like to obtain a description of X whose "complexity" only depends on K. (Often, one additionally requires $X = X^{-1}$ and $1 \in X$.) The oldest result of this type, dating from the 60ies, is due to Freiman himself and describes such sets X in $G = \mathbb{Z}$. Later, there have been several generalizations until finally, in 2009, Breuillard-Green-Tao obtained a result in arbitrary groups G. It turns out that if G is non-commutative, then to obtain good descriptions of X, it is better to impose a somewhat stronger condition on X than just $|X \cdot X| \leq K|X|$: either one requires that $X \cdot X$ can be covered by a fixed number of (say, left-) translates of X (such an X is called a *K-approximate subgroup*), or one requires that $|X \cdot X \cdot X| \leq K|X|$ for some fixed K. These two conditions are sufficiently similar that for results like the one of Breuillard-Green-Tao, it doesn't matter which one is used.

There are two types of basic examples of approximate subgroups. One of them are actual subgroups of G. For the other one, suppose first that we have elements $a_1, \dots, a_n \in G$ such that the generated group $G' := \langle a_1, \dots a_d \rangle$ is commutative. Moreover, choose $n_1, \dots, n_d \in \mathbb{N}$ and consider $P := \{a_1^{r_1} \cdots a_d^{r_d} \mid -n_i \leq r_i \leq n_i\}$. This set is a K-approximate subgroup where K only depends on d (one can take $K = 3^d$). If the generated group G' is nilpotent, then by imposing some additional conditions on the a_i and r_i, one also obtains that P is an approximate subgroup; such P are called *nilprogression*.

The two above kinds of examples can be combined: if we have $H \triangleleft G' \subseteq G$ with G'/H nilpotent and P' is a nilprogression in G'/H, then its preimage P in G is also an approximate subgroup. The result of Breuillard-Green-Tao says that any approximate subgroup X "is close to" a P' of this form, where the d appearing in the definition of P' only depends on K (and not on X or G). There are different possibilities to define what it means for X and P' to be close to each other; for example, one can require that $X \cdot X$ and P' are *C-commensurable* for some C depending only on K: each of the two sets $X \cdot X$ and P' can be covered by C-many translates of the other one.

In the talk, I sketched a model theoretic variant of the proof of Breuillard-Green-Tao; this variant originates from lecture notes of Hrushovski which can be found on his web page. The first step is to reformulate the result using ultraproducts; the advantage of this is that one gets rid of all the conditions about contants depending only on K. More precisely, one assumes that G is a ultraproduct of groups G_i and $X \subseteq G$ is an ultraproduct of K-approximate subgroups $X_i \subseteq G_i$ (all for the same K). The claim is that then, there exist definable G', H, P' satisfying conditions similar to the classical result for some $d, C \in \mathbb{N}$.

The main steps of the proof is the ultraproduct version of the result are the following:

- The result [add reference here] from the talk Approximate Subgroups I can be applied to X and yields a $\wedge$-definable group S.

- Set $\tilde{G} := \langle X \rangle$. Using that the quotient $L := \tilde{G}/S$ is bounded, one obtains that it is a locally compact group. By a theorem of Gleason-Yamabe, L has a subquotient which is a Lie-group, so by making $\tilde{G}$ smaller and S larger, we can assume that L itself is a Lie-group.

- The central part of the proof is then to show that L is nilpotent. To this end, one defines a "distance to 1" in $\tilde{G}$. Then, one the one hand, the commutator of two elements close to 1 is even closer to 1 (this is certainly true in Lie groups, and part of the work is to show that it is also true in $\tilde{G}$). On the other hand, using that X is pseudo-finite, we can find an element $a \neq 1$ that has mininimal distance to 1. This implies that a is central and by repeating this argument (and using an induction over $\dim L$), one obtains that L is nilpotent.

Isometries in valued fields

To any definable set $X \subseteq \mathbb{Z}_p^n$ in the p-adic inteters, one can associate the "Poincaré series" $P_X(t) \in \mathbb{Q}[[t]]$, a formal power series which, for each $\lambda \in \mathbb{N}$, counts the points of the image of X in $(\mathbb{Z}/p^\lambda\mathbb{Z})^n$. In the 80ies, Denef proved that this series is a rational function, i.e. $P_X(t) \in \mathbb{Q}(t)$. Since $P_X(t)$ depends only on the isometry type of X (with respect to the ultrametric maximum norm on $\mathbb{Z}_p^n$), this result can be regarded as a (partial) description of definable sets up to isometry. The goal of this talk was to present a more geometric such description, which in particular implies the result of Denef. More precisely, our main theorem describes definable sets up to definable isometry in Henselian valued fields K of characteristic $(0,0)$; using the usual compactness argument, this then yields a result in $\mathbb{Q}_p$ for p sufficiently big, which in turn implies Denef's result.

The main result in K roughly is the following. Given a definable set $X \subseteq K^n$, there exists a definable partition $K^n = S_0 \,\dot\cup \ldots \dot\cup\, S_n$, where S_i is of dimension i, such that X is "d-translatable away from $S_0 \cup \cdots \cup S_{d-1}$". More precisely, for any ball $B \subseteq K^n$ disjoint from $S_0 \cup \cdots \cup S_{d-1}$, $X \cap B$ is, up to isometry, of the form $B_0 \times X'$ where B_0 is a d-dimensional ball and $X' \subseteq K^{n-d}$. (In fact, the precise result is even a bit stronger than just up to isometry.)

This result in K resembles results from real and complex geometry about the existence of "Whitney stratifications", which describe singularities of subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$. In fact, by applying our result to the case where K is an elementary extension of $\mathbb{R}$ or $\mathbb{C}$ (which is, in a natural way, a Henselian valued field), one can even check that the valued field result implies the existence of Whitney stratifications. (This was not part of the talk.)

The corresponding article ("Non-Archimedean Whitney stratifications") can be found on `http://arxiv.org/abs/1109.5886`.